Axis-Crossings of the Phase of Sine Wave Plus Noise

By A. J. RAINAL

(Manuscript received November 16, 1966)

This paper is concerned with the axis-crossings of the resultant phase $-\pi \leqq \theta(t,a) \leqq \pi$ of a sinusoidal signal of amplitude $\sqrt{2a}$ and frequency f_0 plus Gaussian noise of unit variance having a narrow-band power spectral density which is symmetrical about f_0. The discontinuous phase process $\theta(t,a)$ is present at the output of the IF amplifier of a radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise. Also, the phase process $\theta(t,a)$ is basic in Rice's recent analysis of noise in FM receivers. The following theoretical results are presented concerning the axis-crossings (level-crossings) of $\theta(t,a)$ at an arbitrary level θ:

(i) The average number of upward (or downward) axis-crossings per second.

(ii) The conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t.

(iii) The conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

(iv) The conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

(v) The variance of the number of axis-crossings observed in a time τ.

The theoretical probability functions are presented in graphs as a continuous function of τ for various values of θ and "a" for the case when the Gaussian noise has a Gaussian power spectral density.

I. INTRODUCTION

Consider the stationary random process $I(t,a)$ consisting of a sinusoidal signal of amplitude $\sqrt{2a}$ and frequency f_0 plus Gaussian noise $I_N(t)$, of unit variance, having a narrow-band power spectral density $W_b(f - f_0)$ which is symmetric about f_0. Rice's[1] graphical representation of $I(t,a)$ is illustrated in Fig. 1 in order to define the Rayleigh envelope process $R(t,a)$ and the resultant phase $-\pi \leqq \theta(t,a) \leqq \pi$. The purpose of this

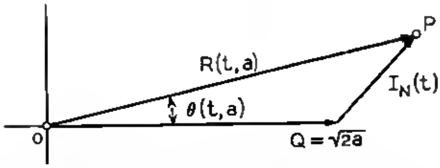

Fig. 1 — Graphical representation of $I(t,a) = \sqrt{2a} \cos 2\pi f_0 t + I_N(t) = R(t,a) \cos [2\pi f_0 t + \theta(t,a)]$. The point P wanders around, as time goes by, in the plane of the figure and generates the phase process $\theta(t,a)$.

paper is to present some theoretical results concerning the axis-crossing points of the stationary, discontinuous phase process $\theta(t,a)$. In the literature, these same points are also called level-crossings. The axis-crossing points and the axis-crossing intervals of $\theta(t,a)$ are defined in Fig. 2. The axis-crossing points and the axis-crossing intervals of $R(t,a)$ are defined in a similar manner and were discussed by Rice[2] and Rainal.[3,4] The Rayleigh process $R(t,a)$ and the phase process $\theta(t,a)$ are present at the output of the IF amplifier of a typical radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise. Also, the phase process $\theta(t,a)$ is basic in Rice's[5] recent analysis of noise in FM receivers.

Using a notation consistent with Refs. 3 and 6, we shall present the following theoretical results, in terms of well-known tabulated functions, concerning the axis-crossings of $\theta(t,a)$ at an arbitrary level θ and arbitrary signal-to-noise power ratio "a":

(*i*) N_θ, the average number of upward (or downward) axis-crossings per second.

(*ii*) $Q_1^-(\tau,\theta,a)\, d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t.

(*iii*) $Q_1^+(\tau,\theta,a)\, d\tau$, the conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

(*iv*) $[U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)]\, d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

II. AVERAGE NUMBER OF AXIS-CROSSINGS PER SECOND

N_θ, the average number of upward axis-crossings per second of the level θ by the phase process $\theta(t,a)$, follows directly from some results due to Rice. Rice[1] showed that

$$N_\theta = \int_0^\infty dR \int_0^\infty d\theta' \, \theta' P(R,\theta,\theta'), \tag{1}$$

where

$$P(R,\theta,\theta') = \frac{R^2}{2\pi\sqrt{2\pi\beta}} \exp\left[-\frac{R^2}{2} - \frac{(\theta' R)^2}{2\beta} + QR\cos\theta - \frac{Q^2}{2}\right]$$

$$Q = \sqrt{2a}$$

$$\beta = 4\pi^2 \int_0^\infty W_b(f - f_0)(f - f_0)^2 \, df$$

$$-\pi \leqq \theta \leqq \pi.$$

$W_b(f - f_0)$ = one-sided narrow-band power spectral density of $I_N(t)$. Performing the integrations we find that

$$N_\theta = \frac{\sqrt{\beta}}{2\pi} \exp\left[-a \sin^2 \theta\right] \Phi(Q \cos \theta), \tag{2}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy.$$

Equation (2) was also derived by Tikhonov[7].

Since $\theta = 0$ is a level of symmetry we have that $N_\theta = N_{-\theta}$. Also, the average number of downward axis-crossings per second is given by the right-hand side of (1) with the upper limit of integration of θ' set to $-\infty$. Thus, the average number of downward axis-crossings per second is also equal to N_θ.

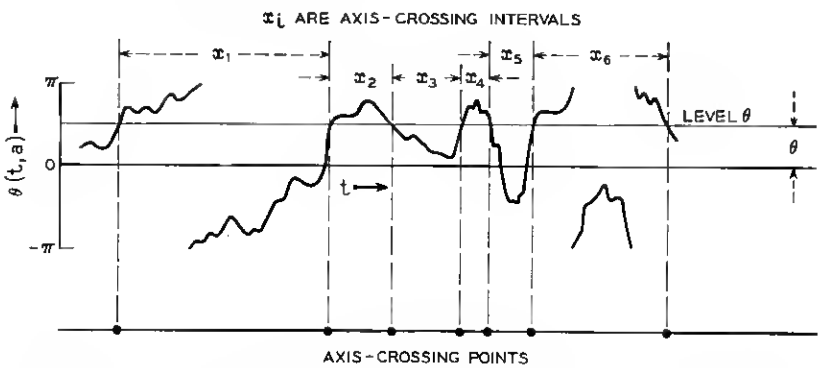

Fig. 2 — The level θ defines the axis-crossing points and the axis-crossing intervals of the discontinuous phase process $\theta(t,a)$.

When the level $\theta = \pm\pi$ and "a" is large, $2N_\theta\tau$ represents the average number of clicks observed in a time τ at the output of an ideal FM receiver[5,8] during the reception of a unmodulated carrier in tbe presence of receiver noise. The variance of the number of clicks observed in a time τ is discussed in Section IV.

III. CONDITIONAL PROBABILITY FUNCTIONS

The reader should refer to Rice[2] for the definition of all notation which is not defined in this paper. For the phase process $\theta(t,a)$, tbe conditional probability $Q_1^-(\tau,\theta,a)\,d\tau$, the conditional probability that an upward axis-crossing of the level θ occurs between $t+\tau$ and $t+\tau+d\tau$ given a downward axis-crossing of the level θ at t, is given by an equation analogous to Rice's[2] equation (86):

$$Q_1^-(\tau,\theta,a)\,d\tau = -d\tau\, N_\theta^{-1}\int_{-\infty}^{\infty} dR_1' \int_{-\infty}^{\infty} dR_2' \int_0^{\infty} dR_1 \int_0^{\infty} dR_2 \cdot \int_{-\infty}^{0} d\theta_1' \int_0^{\infty} d\theta_2'\, \theta_1'\theta_2' p(R_1, R_1', \theta, \theta_1', R_2, R_2', \theta, \theta_2'), \tag{3}$$

where

$$\begin{aligned} &p(R_1, R_1', \theta, \theta_1', R_2, R_2', \theta, \theta_2') \\ &= \frac{R_1^2R_2^2}{(2\pi)^4 M}\exp\left\{-\frac{1}{2M}\Big[M_{11}[R_1^2 + R_2^2 - 2Q(R_1+R_2)\cos\theta + 2Q^2]\right. \\ &\quad + 2M_{12}[R_1R_1' - R_2R_2' - Q(R_1' - R_2')\cos\theta + Q(R_1\theta_1' - R_2\theta_2')\sin\theta] \\ &\quad + 2M_{13}[R_1R_2' - R_2R_1' - Q(R_2' - R_1')\cos\theta + Q(R_2\theta_2' - R_1\theta_1')\sin\theta] \\ &\quad + 2M_{14}[R_1R_2 - Q(R_2 + R_1)\cos\theta + Q^2] \\ &\quad \left. + M_{22}[R_1'^2 + R_2'^2 + R_1^2\theta_1'^2 + R_2^2\theta_2'^2] + 2M_{23}[R_1'R_2' + R_1R_2\theta_1'\theta_2']\Big]\right\}. \end{aligned}$$

The M's are given in Rice's[2] Appendix I with

$$m(\tau) = \int_0^{\infty} W_b(f - f_0)\cos 2\pi(f - f_0)\tau\, df. \tag{4}$$

By performing the integrations with respect to R_1' and R_2' we find that

$$Q_1^-(\tau,\theta,a) = -N_\theta^{-1}\int_0^{\infty} dR_1 \int_0^{\infty} dR_2 \cdot \int_{-\infty}^{0} d\theta_1' \int_0^{\infty} d\theta_2'\, \theta_1'\theta_2' p(R_1, \theta, \theta_1', R_2, \theta, \theta_2'), \tag{5}$$

where

$$p(R_1\,,\ \theta,\ \theta_1'\,,\ R_2\,,\ \theta,\ \theta_2') = \frac{R_1^2R_2^2}{(2\pi)^3\sqrt{M_{22}^2 - M_{23}^2}}$$

$$\cdot \exp\left\{-\frac{1}{2M}\left[M_{22}(R_1^2\theta_1'^2 + R_2^2\theta_2'^2) + 2M_{23}R_1R_2\theta_1'\theta_2'\right.\right.$$

$$\left.\left. + 2Q\sin\theta[M_{12} - M_{13}][R_1\theta_1' - R_2\theta_2']\right]\right\}\cdot \exp(-G_0/2M)$$

and

$$G_0 = \left\{M_{11}(R_1^2 + R_2^2) + 2M_{14}R_1R_2\right.$$

$$+ 2Q(Q - R_1\cos\theta - R_2\cos\theta)(M_{11} + M_{14})$$

$$+ \frac{(-M_{12}^2M_{22} - M_{13}^2M_{22} + 2M_{12}M_{13}M_{23})}{(M_{22}^2 - M_{23}^2)}$$

$$\cdot[(R_1 - Q\cos\theta)^2 + (R_2 - Q\cos\theta)^2]$$

$$+ \frac{(-M_{12}^2M_{23} - M_{13}^2M_{23} + 2M_{12}M_{13}M_{22})}{(M_{22}^2 - M_{23}^2)}$$

$$\left.\cdot[2(R_1 - Q\cos\theta)(R_2 - Q\cos\theta)]\right\}.$$

By introducing the variables x, y, in place of θ_1', θ_2' with the following transformation

$$R_1\theta_1' = -\left[\frac{M_{22}}{1 - m^2}\right]^{\frac{1}{2}}x - Q\left[\frac{M_{12} - M_{13}}{M_{22} - M_{23}}\right]\sin\theta \tag{6}$$

$$R_2\theta_2' = \left[\frac{M_{22}}{1 - m^2}\right]^{\frac{1}{2}}y + Q\left[\frac{M_{12} - M_{13}}{M_{22} - M_{23}}\right]\sin\theta, \tag{7}$$

we find that

$$\begin{aligned} Q_1^-(\tau,\theta,a) &= \frac{N_\theta^{-1}M_{22}}{(2\pi)^2(1 - m^2)^2}J(r_1\,,\ h_1) \\ &\cdot \exp\left\{\frac{Q^2\sin^2\theta(M_{12} - M_{13})^2}{M(M_{22} - M_{23})}\right\}\int_0^\infty dR_1\int_0^\infty dR_2\,\exp(-G_0/2M), \end{aligned} \tag{8}$$

where

$$J(r_1\,,\ h_1) = \frac{1}{2\pi\sqrt{1 - r_1^2}}\int_{h_1}^\infty dx\int_{h_1}^\infty dy\ (x - h_1)(y - h_1)e^z$$

$$z = -\frac{x^2 + y^2 - 2r_1xy}{2(1 - r_1^2)}$$

$$r_1 = \frac{M_{23}}{M_{22}}$$

$$h_1 = -Q\left[\frac{M_{12} - M_{13}}{M_{22} - M_{23}}\right]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}} \sin\,\theta.$$

Finally, by introducing tbe new variables x_0, y_0 in place of R_1, R_2 with the following transformation

$$R_1 = x_0 + Q\cos\theta \tag{9}$$

$$R_2 = y_0 + Q\cos\theta, \tag{10}$$

we find, after some simplifications using Jacobi's[9] theorem, that

$$Q_1^-(\tau,\theta,a) \tag{11}$$
$$= [2\pi N_\theta]^{-1}[1 - m^2]^{-\frac{1}{2}}M_{22}J(r_1, h_1)\exp\left[\frac{-2a\sin^2\theta}{1 + m}\right]K(m, h_0),$$

where

$$K(m, h_0) = \frac{1}{2\pi\sqrt{1 - m^2}}\int_{h_0}^{\infty} dx_0 \int_{h_0}^{\infty} dy_0\, e^{z_0}$$

$$z_0 = -\frac{x_0^2 + y_0^2 - 2mx_0y_0}{2(1 - m^2)}$$

$$h_0 = -Q\cos\theta$$

$$h_1 = -Q\left[\frac{m'}{1 + m}\right]\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}\sin\,\theta.$$

The conditional probability $Q_1^+(\tau,\theta,a)\,d\tau$, the conditional probability that a downward axis-crossing of the level θ occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t, is obtained from (3) by changing the signs of the ∞'s in the limits of integration of θ_1' and θ_2'. We find that $Q_1^+(\tau,\theta,a)$ is equal to the right-hand side of (11) with h_1 replaced by $-h_1$. This latter result also follows from tbe symmetry relation $Q_1^+(\tau,\theta,a) = Q_1^-(\tau,-\theta,a)$.

The conditional probability $[U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)]\,d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t, is obtained from (3) by changing the lower limit of integration of θ_1' to $+\infty$. We find that $U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)$ is equal to the right-hand side of (11) with the

function $J(r_1, h_1)$ replaced by the function $J_1(r_1, h_1)$, where

$$J_1(r_1, h_1) = \frac{1}{2\pi\sqrt{1 - r_1^2}} \int_{h_1}^{-\infty} dx \int_{h_1}^{\infty} dy\, (x - h_1)(y - h_1)e^{z}. \tag{12}$$

The conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t is obtained from (3) by changing the upper limit of integration of θ_2' to $-\infty$. The result is that this conditional probability function is equal to the conditional probability function $U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)$ as one would expect from symmetry.

The functions $J(r_1, h_1)$, $K(m,h_0)$, and $J_1(r_1, h_1)$ are expressed in terms of Karl Pearson's [10,11,12] tabulated function (d/N) in Refs. 2 and 3. Thus, the conditional probability functions $Q_1^-(\tau,\theta,a)$, $Q_1^+(\tau,\theta,a)$, and $U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)$ are expressed in terms of well-known tabulated functions.

Since $\theta = 0$ is a level of symmetry, we need only discuss the conditional probabilities when θ is restricted to the interval $0 \leqq \theta \leqq \pi$. The corresponding results when θ is in the remaining interval $-\pi \leqq \theta < 0$ can be deduced from the following symmetry conditions:

$$Q_1^-(\tau,\theta,a) = Q_1^+(\tau,-\theta,a) \tag{13}$$

$$U_1(\tau,\theta,a) - Q_1(\tau,\theta,a) = U_1(\tau,-\theta,a) - Q_1(\tau,-\theta,a). \tag{14}$$

IV. VARIANCE OF THE NUMBER OF AXIS-CROSSINGS IN A TIME τ

For an arbitrary level θ and arbitrary signal-to-noise ratio "a," let $N(\tau,\theta,a)$ denote the number of axis-crossings observed in a time τ. Then, we have that

$$EN(\tau,\theta,a) = 2N_\theta\tau \tag{15}$$

and

$$\operatorname{Var} N(\tau,\theta,a) \equiv EN^2(\tau,\theta,a) - [2N_\theta\tau]^2, \tag{16}$$

where

$$E = \text{Expectation}$$

$$\text{Var} = \text{Variance}.$$

Using McFadden's [13] general result, also see Rice's derivation in Bendat, [14] we have that

$$EN^2(\tau,\theta,a) = 2N_\theta\tau + 4N_\theta \int_0^\tau (\tau - x)U_1(x,\theta,a)\, dx. \tag{17}$$

In this latter equation, $U_1(\tau,\theta,a)\,d\tau$ denotes the conditional probability that an axis-crossing occurs between $t+\tau$ and $t+\tau+d\tau$ given an axis-crossing at time t. Since the joint probability that an axis-crossing occurs between $t+\tau$ and $t+\tau+d\tau$ and an axis-crossing occurs between t and $t+dt$ can be expressed as

$$\begin{aligned} 2N_\theta U_1(\tau,\theta,a)\,dt\,d\tau &= 2N_\theta[U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)]\,dt\,d\tau \\ &\quad + N_\theta Q_1^+(\tau,\theta,a)\,dt\,d\tau + N_\theta Q_1^-(\tau,\theta,a)\,dt\,d\tau, \end{aligned} \tag{18}$$

we have that

$$U_1(\tau,\theta,a) = [U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)] + \tfrac{1}{2}Q_1^+(\tau,\theta,a) + \tfrac{1}{2}Q_1^-(\tau,\theta,a). \tag{19}$$

Thus, Var $N(\tau,\theta,a)$ can be computed by using (16), (2), (17), and (19)

$$\text{Var}\,N(\tau,\theta,a) = 2N_\theta\tau + 4N_\theta\int_0^\tau (\tau - x)U_1(x,\theta,a)\,dx - [2N_\theta\tau]^2 \tag{20}$$

$$= 2N_\theta\tau\left\{1 + 2\int_0^\tau\left[1 - \frac{x}{\tau}\right][U_1(x,\theta,a) - 2N_\theta]\,dx\right\}. \tag{21}$$

For large τ, (21) becomes

$$\text{Var}\,N(\tau,\theta,a) \doteq 2N_\theta\tau\left\{1 + 2\int_0^\tau [U_1(x,\theta,a) - 2N_\theta]\,dx\right\}. \tag{22}$$

When twice the value of the integral in (22) is small compared with unity we have that

$$\text{Var}\,N(\tau,\theta,a) \doteq 2N_\theta\tau. \tag{23}$$

This is the relation one would expect if the axis-crossing points represent a poisson point process for whicb $U_1(\tau,\theta,a) \equiv 2N_\theta$ for all τ.

Rice[5] assumed a poisson point process for the case $\theta = \pi$ and "a" large in order to use (23) in his analysis of noise in FM receivers. Indeed, for the case of a Gaussian autocorrelation function (22) serves to justify Rice's use of (23) for large τ, $\theta = \pi$, and $a \geqq 4$. For this case, with $a = 4$, numerical integration showed that the value of the integral in (22) is approximately 0.05.

Notice that (22) not only applies to the point process defined by $\theta(t,a)$ but also applies to more general stationary point processes.

Incidentally, the probability function $U_1(\tau,\theta,a)$ can also be used to compute, approximately, the probability density $p_0(\tau,\theta,a)$ of the axis-crossing intervals x_i by using the following basic integral equation of renewal theory:

$$p_0(\tau,\theta,a) = U_1(\tau,\theta,a) - p_0(\tau,\theta,a) * U_1(\tau,\theta,a). \tag{24}$$

The symbol $*$ denotes the convolution operation, that is,

$$f * g \equiv \int_{-\infty}^{\infty} f(t)g(\tau - t)\,dt. \tag{25}$$

Equation (24) is based on the assumption that a given axis-crossing interval is statistically independent of the sum of the previous $(m + 1)$ axis-crossing intervals for all non-negative integral m. A theorem in Paragraph 5.2 shows that the assumption is false when $m = 0$. Thus, (24) can only yield approximate results.

The exact probability density of the axis-crossing intervals x_i is at present unknown. However, the first moment of this probability density is equal to $[2N_\theta]^{-1}$.

V. SOME SPECIAL CASES AND A THEOREM

In this section we shall state some special cases of tbe conditional probability functions. We shall also present a theorem concerning the dependence of two successive axis-crossing intervals.

5.1 *Large τ and Fixed θ, a*

As τ becomes large we find that $Q_1^-(\tau,\theta,a)$, $Q_1^+(\tau,\theta,a)$, and $U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)$ approach the value N_θ as one would expect.

5.2 *Small τ and Fixed θ, a*

By expanding $m(\tau)$ as

$$m(\tau) = 1 - \frac{\beta}{2}\tau^2 + \frac{b_3 \mid \tau^3 \mid}{3!} + \frac{b_4\tau^4}{4!} + \frac{b_5 \mid \tau^5 \mid}{5!} + \cdots, \tag{26}$$

we find that as $\tau \to 0$ from the right with $b_3 \neq 0$

$$Q_1^-(\tau,\theta,a) \to Q_1^+(\tau,\theta,a) \to \frac{2b_3}{3\beta}\left[\frac{3\sqrt{3} + 2\pi}{12\pi}\right] \tag{27}$$

$$U_1(\tau,\theta,a) - Q_1(\tau,\theta,a) \to \frac{2b_3}{3\beta}\left[\frac{3\sqrt{3} - \pi}{12\pi}\right]. \tag{28}$$

Equation (28) suggests that wiggles having infinite rapidity and infinitesimal amplitude are associated with the phase process $\theta(t,a)$ when $b_3 \neq 0$ or $W_b(f - f_0) = O(f^{-4})$ as $f \to \infty$.

We also find that for small τ with $b_3 = 0$:

$$Q_1^-(\tau,\theta,a) \doteq \frac{b_4 - \beta^2}{4\beta} J(1,h_1)\tau \exp\left[-\frac{\beta}{8}(\tau Q \sin\theta)^2\right] \tag{29}$$

$$Q_1^+(\tau,\theta,a) \doteq \frac{b_4 - \beta^2}{4\beta} J(1,-h_1)\tau \exp\left[-\frac{\beta}{8}(\tau Q \sin\theta)^2\right] \quad (30)$$

$$U_1(\tau,\theta,a) - Q_1(\tau,\theta,a) \doteq \frac{b_4 - \beta^2}{4\beta} J_1(r_1, h_1)\tau \exp\left[-\frac{\beta}{8}(\tau Q \sin\theta)^2\right], \quad (31)$$

where

$$h_1 \doteq \frac{\beta Q \sin\theta}{\sqrt{b_4 - \beta^2}}.$$

It is interesting to compare the above results with the corresponding results at the level I of a Gaussian process $I(t)$ having the normalized autocorrelation function $m(\tau)$. That is, compare the above results with Rice's[2] equation (63) or Rainal's[6] equations (44), (52), (53), and (54) when $I = Q \sin\theta$. The results are identical.

Thus, a theorem[6] concerning the dependence of two successive axis-crossing intervals of the Gaussian process $I(t)$ also applies to the phase process $\theta(t,a)$. That is, if $\theta(t,a)$ is a phase process, defined in paragraph one, having a finite expected number of axis-crossing points per unit time at any level θ, then two successive axis-crossing intervals at that level θ are statistically dependent.

The theorem implies that successive axis-crossing points do not form a Markov or Poisson point process.

5.3 $Q_1^+(\tau,\theta,a)$ *for small* τ, $b_3 = 0$, *and large* $Q \sin\theta$

For small τ and large $Q \sin\theta$ with $b_3 = 0$ or $W_b(f - f_0) \neq O(f^{-4})$ as $f \to \infty$, we find from (30) that

$$Q_1^+(\tau,\theta,a) \doteq \frac{\beta}{4}\tau(Q\sin\theta)^2 \exp\left[-\frac{\beta}{8}(\tau Q\sin\theta)^2\right]. \quad (32)$$

Thus, $Q_1^+(\tau,\theta,a)$ is approximated by a Rayleigh probability density identical to Rice's[2] equation (65) when $I = Q \sin\theta$.

5.4 $a = 0$ *and arbitrary* θ, τ

When $a = 0$ we find that

$$\begin{aligned} Q_1^-(\tau,\theta,0) &= Q_1^+(\tau,\theta,0) \\ &= 2\beta^{-\frac{1}{2}}[1 - m^2]^{-\frac{3}{2}} \frac{M_{22}}{(2\pi)^2}[r_1(\pi - \cos^{-1} r_1) \qquad (33) \\ &\qquad + \sqrt{1 - r_1^2}][\pi - \cos^{-1} m] \end{aligned}$$

$$U_1(\tau,\theta,0) - Q_1(\tau,\theta,0) = 2\beta^{-\frac{1}{2}}[1 - m^2]^{-\frac{3}{2}} \frac{M_{22}}{(2\pi)^2} [-r_1 \cos^{-1} r_1 + \sqrt{1 - r_1^2}][\pi - \cos^{-1} m], \tag{34}$$

where

$$0 \leqq \cos^{-1} r_1 \leqq \pi$$

$$0 \leqq \cos^{-1} m \leqq \pi.$$

Thus, when $a = 0$ the conditional probabilities are independent of the level θ as one would expect.

5.5 *Large a, $\theta = 0$, and arbitrary τ*

When "a" is large and $\theta = 0$ we find that

$$\begin{aligned} Q_1^-(\tau,0,a) &= Q_1^+(\tau,0,a) \\ &\doteq \beta^{-\frac{1}{2}}[1 - m^2]^{-\frac{1}{2}} \frac{M_{22}}{2\pi} [r_1(\pi - \cos^{-1} r_1) + \sqrt{1 - r_1^2}] \end{aligned} \tag{35}$$

$$\begin{aligned} U_1(\tau,0,a) &- Q_1(\tau,0,a) \\ &\doteq \beta^{-\frac{1}{2}}[1 - m^2]^{-\frac{1}{2}} \frac{M_{22}}{2\pi} [-r_1 \cos^{-1} r_1 + \sqrt{1 - r_1^2}]. \end{aligned} \tag{36}$$

Thus, when "a" is large and $\theta = 0$, the conditional probabilities are independent of "a".

Again, it is interesting to compare the above results with the corresponding results at the level $I = 0$ of a Gaussian process $I(t)$ having the normalized autocorrelation function $m(\tau)$. That is compare the above results with Rice's[2] equations (62) and (85a). The results are identical. One would expect identical results from Rice's[1] equation (3.6).

5.6 *$\theta = \pi$ and arbitrary a, τ*

When $\theta = \pi$ we find that

$$\begin{aligned} Q_1^-(\tau,\pi,a) &= Q_1^+(\tau,\pi,a) \\ &= [\beta^{\frac{1}{2}}(1 - m^2)^{\frac{1}{2}}\Phi(-Q)]^{-1} \frac{M_{22}}{2\pi} [r_1(\pi - \cos^{-1} r_1) \\ &\qquad + \sqrt{1 - r_1^2}]K(m,Q) \end{aligned} \tag{37}$$

$$U_1(\tau,\pi,a) - Q_1(\tau,\pi,a)$$
$$= [\beta^{\frac{1}{2}}(1-m^2)^{\frac{1}{2}}\Phi(-Q)]^{-1} \frac{M_{22}}{2\pi} [-r_1 \cos^{-1} r_1 + \sqrt{1-r_1^2}]K(m,Q). \quad (38)$$

VI. RESULTS FOR A GAUSSIAN AUTOCORRELATION FUNCTION

For purposes of computation we shall take $W_b(f - f_0)$ and $m(\tau)$ as follows:

$$W_b(f - f_0) = \frac{1}{\sigma_b\sqrt{2\pi}} \exp\left[\frac{-(f-f_0)^2}{2\sigma_b^2}\right] \quad (39)$$

and

$$m(\tau) = \exp\left[\frac{-(2\pi\sigma_b\tau)^2}{2}\right]. \quad (40)$$

This particular selection was also made by Rice[2] and Rainal[4] in their study of the duration of fades associated with the Rayleigh process $R(t,a)$.

From (40) we see that it is convenient to define normalized time as $u_b = 2\pi\sigma_b\tau$. All our results are plotted with respect to normalized time u_b. The units of N_θ are now "crossings per unit of normalized time."

Figs. 3 through 11 present the resulting conditional probability functions for various values of the level θ and for various values of signal-to-noise power ratio "a". For large values of u_b all of the conditional probability functions approach the value of N_θ in accordance with Paragraph 5.1.

Figs. 9 and 11 compare $Q^+(\tau,\theta,a)$ for $\theta = \pi/2$ and $a = 4$, 10 with a corresponding Rayleigh density in accordance with (32). Thus, we conclude that the Rayleigh probability density is a good approximation when τ is small and $Q \sin\theta = \sqrt{2a} \sin\theta \geqq 2\sqrt{2}$.

Fig. 7 compares well with Figs. 2 and 3 of Ref. 6. Thus, we conclude that (35) and (36) are good approximations when $a \geqq 4$.

VII. CONCLUSIONS

The theoretical probability functions $Q_1^-(\tau,\theta,a)$, $Q_1^+(\tau,\theta,a)$, and $U_1(\tau,\theta,a) - Q_1(\tau,\theta,a)$ are expressible in terms of well-known tabulated functions. These results can be used to compute Var $N(\tau,\theta,a)$, the variance of the number of axis-crossing points observed in a time τ. These results can also be used to compute, approximately, the probability density of axis-crossing intervals x_i via renewal theory. The exact probability density is at present unknown.

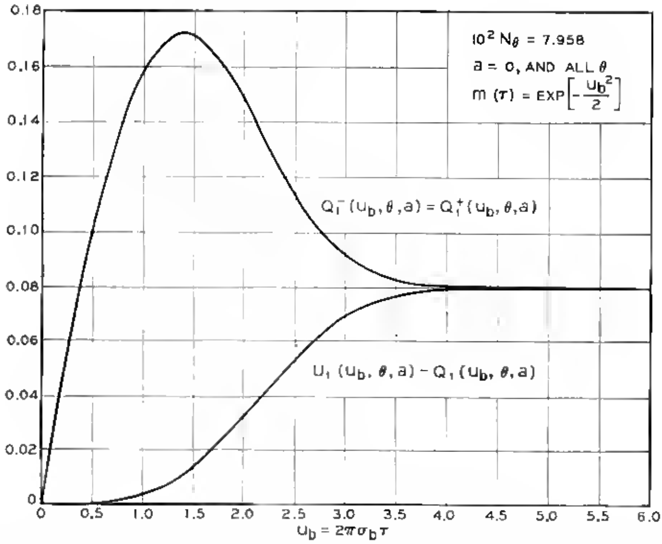

Fig. 3 — Plots of the probability functions $Q_1^-(u_b, \theta, a)$, $Q_1^+(u_b, \theta, a)$ and $U_1(u_b, \theta, a) - Q_1(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

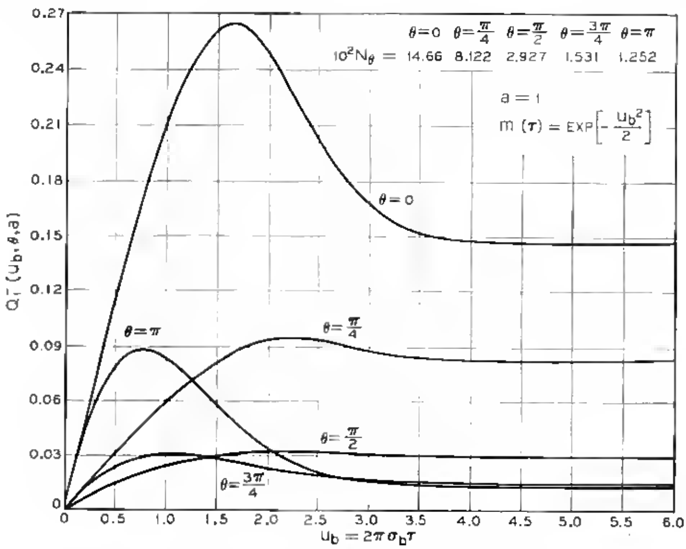

Fig. 4 — Plots of the probability function $Q_1^-(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

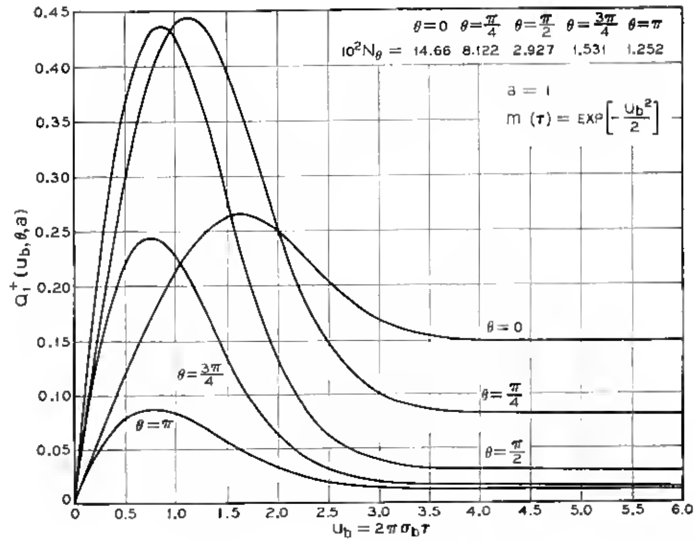

Fig. 5 — Plots of the probability function $Q_1^+(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

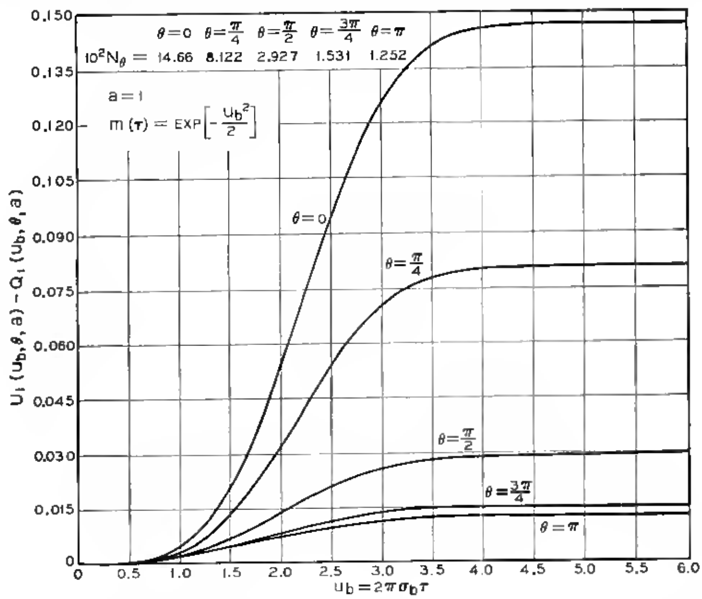

Fig. 6 — Plots of the probability function $U_1(u_b, \theta, a) - Q_1(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

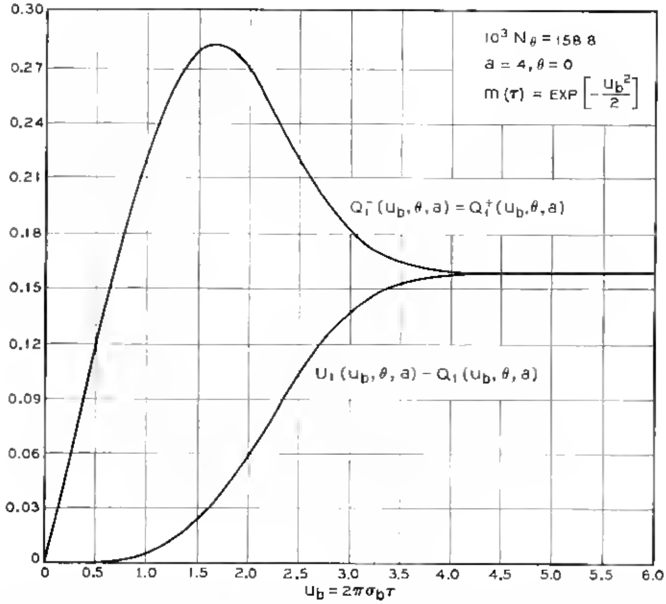

Fig. 7 — Plots of the probability functions $Q_1^-(u_b, \theta, a)$, $Q_1^+(u_b, \theta, a)$ and $U_1(u_b, \theta, a) - Q_1(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

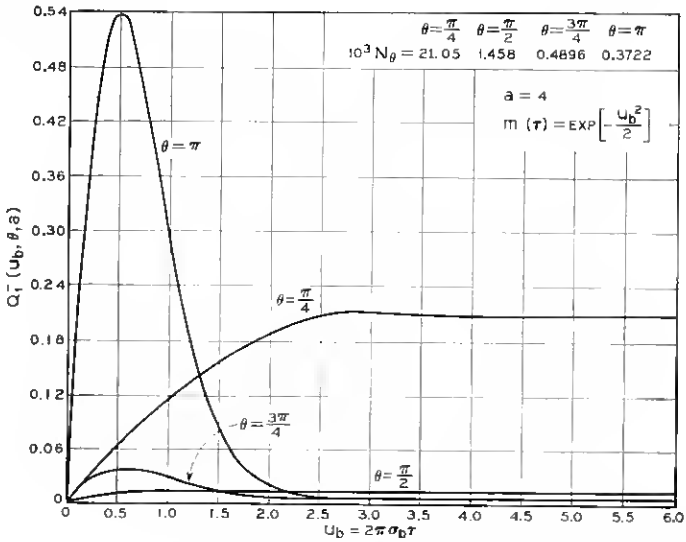

Fig. 8 — Plots of the probability function $Q_1^-(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

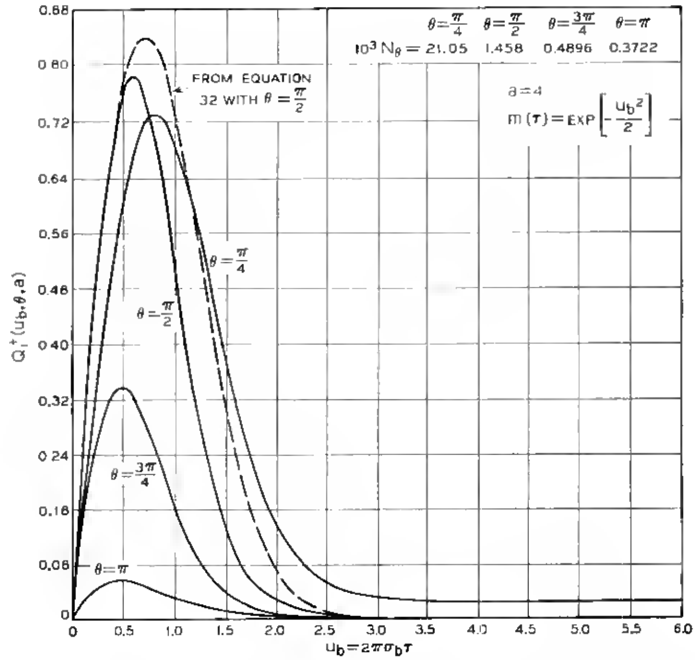

Fig. 9 — Plots of the probability function $Q_1^+(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

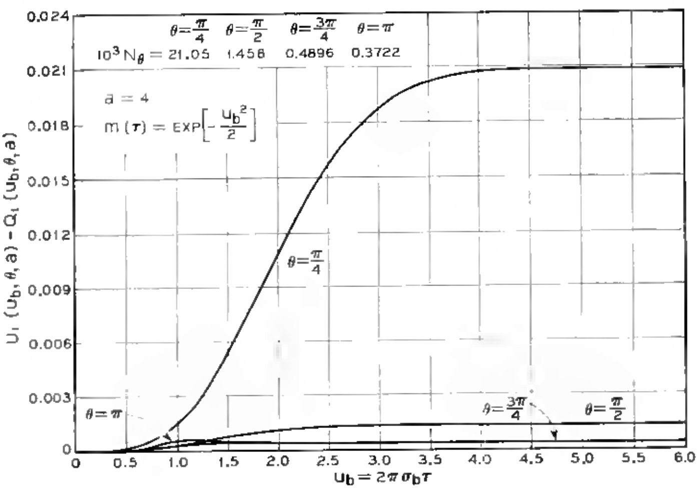

Fig. 10 — Plots of the probability function $U_1(u_b, \theta, a) - Q_1(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

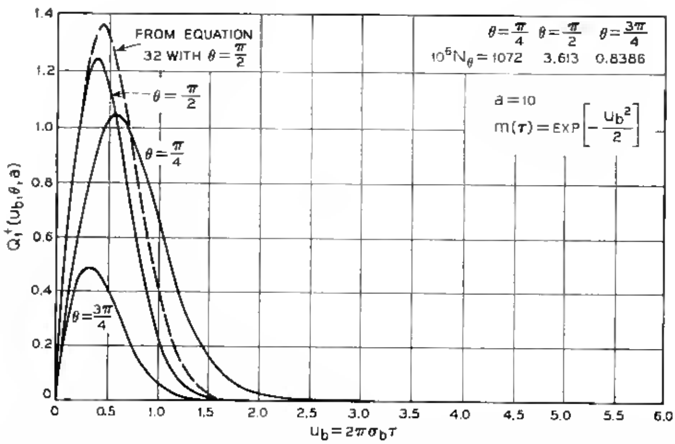

Fig. 11 — Plots of the probability function $Q_1^+(u_b, \theta, a)$ associated with the axis-crossing points defined by the level θ of $\theta(t,a)$. $\theta(t,a)$ denotes the resultant phase of a sinusoidal signal plus narrow-band Gaussian noise having autocorrelation function $m(\tau)$. "a" denotes the signal-to-noise power ratio.

Because the level $\theta = 0$ is a level of symmetry, results for $0 \leqq \theta \leqq \pi$ imply results for $-\pi \leqq \theta < 0$.

When $W_b(f - f_0) = O(f^{-4})$ as $f \to \infty$, wiggles having infinite rapidity and infinitesimal amplitude are associated with the phase process $\theta(t,a)$.

When $\theta = 0$ with the signal-to-noise power ratio $a \geqq 4$, the conditional probability functions associated with the phase process $\theta(t,a)$ are equal, approximately, to the corresponding results for a certain Gaussian process.

When $W_b(f - f_0) \neq O(f^{-4})$ as $f \to \infty$, and $Q \sin \theta$ is large, $Q_1^+(\tau,\theta,a)$ for small τ is approximated by a Rayleigh probability density.

When $\theta = \pi$ and $a \geqq 4$, Var $N(\tau,\theta,a)$ for large τ is equal, approximately, to $2N_\theta\tau$, the variance resulting from a poisson point process.

When N_θ is finite, two successive axis-crossing intervals of $\theta(t,a)$ are statistically dependent. Thus, the axis-crossing points do not represent exactly a Markov point process or a poisson point process.

VIII. ACKNOWLEDGMENT

It gives me great pleasure to acknowledge stimulating discussions with S. O. Rice. I am also indebted to R. T. Piotrowski for programming the digital computer.

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